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New lattice approaches to the $\Delta I = 1/2$ rule

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ABSTRACT

Lattice QCD should allow a derivation of the $\Delta I = 1/2$ rule from first principles, but numerical calculations to date have been plagued by a variety of problems. After a brief review of these problems, we present several new methods for calculating $K \to \pi\pi$ amplitudes. These are designed for Wilson fermions, though they can be used also with staggered fermions. They all involve a non-perturbative determination of matching coefficients. We show how problems of operator mixing can be greatly reduced by using point-split hadronic currents, and how CP violating parts of the $K \to \pi\pi$ amplitudes can be calculated by introducing a fake top quark. Many of the methods can also be applied to the calculation of two body non-leptonic B-meson decays.

1 Introduction

One of the least well understood features of hadronic physics is the $\Delta I = 1/2$ rule in non-leptonic kaon decays. Decays in which isospin changes by $\Delta I = 1/2$ are greatly enhanced over those with $\Delta I = 3/2$. The particular example we focus on here is the ratio of amplitudes for $K \to \pi\pi$ decays:

$$\frac{\mathcal{A}(K \to \pi\pi[I=0])}{\mathcal{A}(K \to \pi\pi[I=2])} \approx 22\,. \tag{1}$$

Although the origin of this large enhancement is not well understood, we do know that, in a QCD-based explanation, most of the enhancement must come from long distance, non-perturbative physics. This is because the contribution from scales where perturbation theory is reliable, say $p \gtrsim 2$ GeV, is known to enhance the $\Delta I = 1/2$ amplitude by only a factor of about two. Attempts to understand the remainder of the enhancement using models of non-perturbative physics have had partial success [1], but we are far from having a convincing demonstration that QCD does explain the $\Delta I = 1/2$ rule.

In principle, lattice QCD is well suited to study this issue [2]. Indeed, practical approaches have been developed for both Wilson [3, 4] and staggered fermions [5, 6]. The problem is that these methods have not yet yielded useful results. Indirect methods, based on using $K \to \pi$ and $K \to$vacuum amplitudes, have large statistical errors [7, 8], while direct calculations of $K \to \pi\pi$ amplitudes have been done at unphysically large quark masses, for which the presence of a scalar resonance may distort the two pion signal [7, 9, 10]. There has been little work on the problem in recent years.

In this paper we revisit the lattice approach for the case of Wilson fermions (or $O(a)$ improved versions thereof). We reevaluate the existing methods, and propose a variety of new approaches. These vary from a method of comparable simplicity to that of Bernard *et al.* [4], to more speculative ideas that are likely to require much smaller lattice spacings than those presently available. The latter are needed to calculate the CP violating parts of the $K \to \pi\pi$ amplitudes. We also reappraise the calculation of $K \to \pi$ matrix elements in the light of the recent progress made in non-perturbative renormalization techniques.

Existing methods rely on lowest order chiral perturbation theory to connect the unphysical amplitudes which are calculated on the lattice to the desired physical amplitude. One advantage of some of our new methods is that they are not dependent on chiral perturbation theory. This means that they can, in principle, also be used to study B-meson decays.

The outline of this paper is as follows. In the next section we summarise the problems faced by lattice calculations, and then, in sec. 3, recall how these are avoided by the direct method of Bernard *el al.* [4]. With the stage thus set, in sec. 4 we then present the first three of our new methods, which also involve the calculation of $K \to \pi\pi$ amplitudes. This is followed in sec. 5 by a reappraisal of the indirect method using $K \to \pi$ amplitudes. In sec. 6, we present a more speculative method for studying the $\Delta I = 1/2$ rule, which is based on the short distance expansion of the T-product of two weak currents. Finally in sec. 7 we suggest a possible idea for a non-perturbative determination of the CP-violating part of the $\Delta S = 1$ weak Hamiltonian by introducing a fictitious top quark. Section 8 contains our conclusions.

2 The problem

In this section we briefly review the source of the difficulty in calculating $\mathcal{A}(K \to \pi\pi)$ with Wilson fermions. Further details can be found in refs. [3, 11] and references therein.

For scales below $M_{_W}$, but above the charm quark mass, the $\Delta S = 1$ part of the effective weak Hamiltonian can be written as

$$\mathcal{H}_{\rm eff}^{\Delta S=1} = \lambda_u \frac{G_F}{\sqrt{2}} \left[C_+(\mu, M_{_W}) O^{(+)}(\mu) + C_-(\mu, M_{_W}) O^{(-)}(\mu) \right] , \tag{2}$$

$$O^{(\pm)} = \left[(\bar{s}\gamma_\mu^L d)(\bar{u}\gamma_\mu^L u) \pm (\bar{s}\gamma_\mu^L u)(\bar{u}\gamma_\mu^L d) \right] - [u \leftrightarrow c] , \tag{3}$$

where $\gamma_\mu^L = \gamma_\mu(1-\gamma_5)/2$ and $\lambda_u = V_{ud}V_{us}^*$. Here and in the following we use the Euclidean metric. We are ignoring for the moment the contribution which arises when the top quark is integrated out. This is suppressed by λ_t/λ_u, where $\lambda_t = V_{td}V_{ts}^*$, and, in the CP conserving sector, makes a very small contribution to the decay amplitudes.

The operators $O^{(\pm)}$ have different transformation properties under isospin. In particular, $O^{(-)}$ is pure $I = 1/2$, whereas $O^{(+)}$ contains parts having both $I = 1/2$ and $I = 3/2$. An explanation of the $\Delta I = 1/2$ rule thus requires that the $K \to \pi\pi$ matrix element of $C_-O^{(-)}$ be substantially enhanced compared to that of $C_+O^{(+)}$. The short distance, perturbative part of the enhancement is that provided by the ratio of Wilson coefficients, C_-/C_+, while the long distance, non-perturbative contribution comes from the ratio of matrix elements of the operators. Consider first the short distance contribution. At $\mu = M_{_W}$, the two Wilson coefficients have nearly equal magnitudes, $|C_-/C_+| = 1 + O[\alpha_s(M_{_W})]$. The enhancement arises from the renormalization group evolution down to $\mu \sim 2$ GeV, at which scale one finds $|C_-/C_+| \approx 2$. This factor is too small by an order of magnitude to explain the $\Delta I = 1/2$ rule.

The remainder of the enhancement must come from the matrix elements of the operators, and these are the quantities that we wish to calculate on the lattice. In fig. 1 we show the Wick contractions that contribute to such matrix elements. The only part of $\mathcal{H}_{\rm eff}^{\Delta S=1}$ which gives rise to $\Delta I = 3/2$ transitions is the $I = 3/2$ part of $O^{(+)}$, and for this operator only contractions (A) and (C) contribute. In contrast, all four contractions are non-vanishing for the $I = 1/2$ parts of $O^{(\pm)}$. Thus, in order to reproduce the $\Delta I = 1/2$ rule, the sum of the contributions of (B) and (D) must be an order of magnitude larger than those of (A) and (C). In fact, as discussed below, contractions (C) and (D) give non-leading contributions in chiral perturbation theory compared to (A) and (B). Thus we expect contraction (B) to be the "source" of the $\Delta I = 1/2$ rule.

There are two major difficulties which arise in the calculation of these diagrams using lattice QCD.

1. Decay amplitudes into two or more particles cannot be calculated directly in Euclidean space. This follows from the theorem of Maiani and Testa [12]. One must, instead, use a model to analytically continue correlations functions from Euclidean to physical momenta. The only exception is for final state particles at rest relative to each other, in which case there is no phase generated by final state interactions. This problem afflicts both $\Delta I = 1/2$ and $3/2$ amplitudes, although it is likely to be worse for the former since final state interactions are stronger for two pions having $I = 0$ [13].

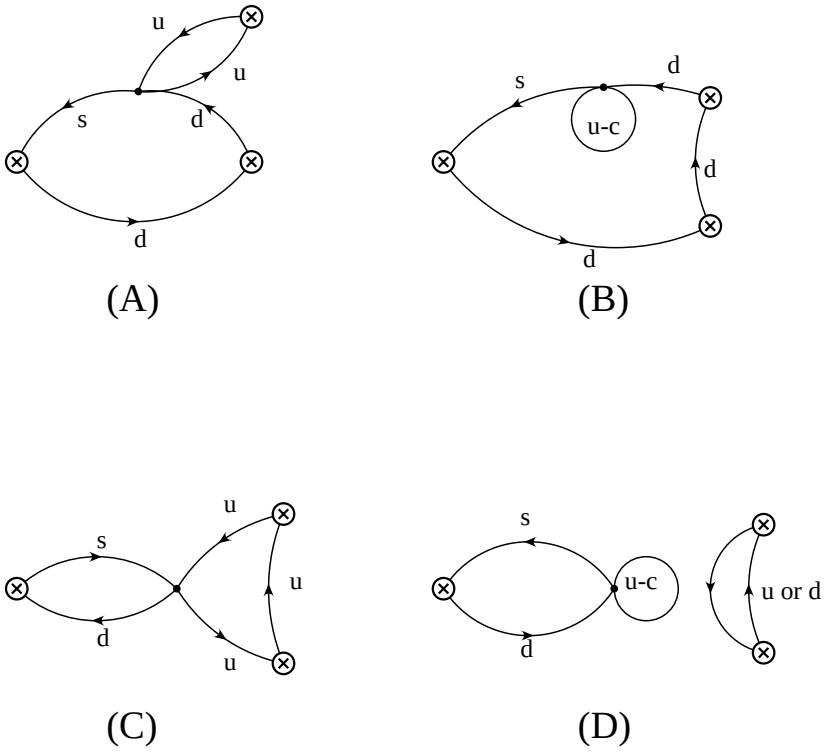

Figure 1: *Contractions contributing to the $\Delta I = 1/2$ amplitude. The lines represent propagators on a background gauge field, and the resulting contraction is to be averaged over gauge configurations with the appropriate measure. The dot represents the operators $O^{(\pm)}$. Different colour contractions are not distinguished.*

2. The "penguin" diagrams (B) and (D) allow the $I = 1/2$ operators to mix with operators of lower dimension with coefficients which diverge as inverse powers of the lattice spacing. This mixing leads to contributions to the amplitudes which are lattice artifacts, and must be subtracted. One must also account for the mixing with other operators of dimension six, although this is less difficult because the mixing coefficients do not diverge in the continuum limit.

In the remainder of this section we expand upon the latter problem. We consider only the negative parity parts of $O^{(\pm)}$, since the parts with positive parity do not contribute to $K \to \pi\pi$ amplitudes.

Lower dimension operators which can appear on the lattice must have the same flavour and CPS parity as $O^{(\pm)}$. CPS is the transformation obtained by combining CP with $(s \leftrightarrow d)$ interchange [14]. Both $O^{(\pm)}$ have positive CPS parity. These symmetries allow mixing with only two lower dimension operators (aside from operators which vanish by the equations of motion):

$$O_p = (m_s - m_d)\bar{s}\gamma_5 d\,, \tag{4}$$

$$\widetilde{O}_\sigma = g_0(m_s - m_d)\bar{s}\sigma_{\mu\nu}\widetilde{G}_{\mu\nu}d\,, \tag{5}$$

Here g_0 is the bare coupling constant and $\widetilde{G}_{\mu\nu}$ the dual field strength. The factors of $m_s - m_d$ are required by CPS symmetry. At dimension six, CPS forbids mixing with other four-fermion operators, and SU(4) flavour forbids mixing between $O^{(+)}$ and $O^{(-)}$. It is noteworthy

that the allowed mixing for the negative parity parts of $O^{(\pm)}$ is, up to this stage, exactly as in the continuum, despite the fact that chiral symmetry is explicitly broken on the lattice [3].

The mixing with O_p and $\widetilde{O}_\sigma$ is further constrained by the GIM mechanism, which implies that the mixing coefficients vanish identically when $m_c = m_u$. In continuum perturbation theory, chiral symmetry requires that the mixing coefficients are quadratic functions of the quark masses and thus vanish as $m_c^2 - m_u^2$. On the lattice, by contrast, chiral symmetry is broken, and the GIM cancellation gives rise only to the factor $m_c - m_u$. This additional factor is sufficient, however, to make the mixing of $O^{(\pm)}$ with $\widetilde{O}_\sigma$ an effect of $O(a)$. Since in this work we are not attempting to remove $O(a)$ effects from the matrix elements of $O^{(\pm)}$, we do not need to consider mixing with $\widetilde{O}_\sigma$. We therefore arrive at the following form of the renormalized operator

$$O^{(\pm)}(\mu) = Z^{(\pm)}(\mu a, g_0^2)\left[O^{(\pm)}(a) + (m_c - m_u)\frac{C_p^{(\pm)}}{a}O_p(a)\right] + O(a)\,. \tag{6}$$

Here $O^{(\pm)}(a)$ and $O_p(a)$ are bare lattice operators, $C_p^{(\pm)}$ are the mixing coefficients, and $Z^{(\pm)}$ are the renormalization constants which cancel the logarithmic divergence of the bare four-fermion operator. The precise definition of the lattice quark masses in eq. (6) is unimportant, since in all practical methods the entire coefficient of O_p is determined non-perturbatively.

We now show that, in spite of the fact that it multiplies a linear divergence (see eq. (6)), it is sufficient to determine $C_p^{(\pm)}$ with an error of $O(a)$, to obtain the physical amplitudes to the same precision. We first note that, in the continuum, O_p is proportional to the divergence of the axial current, apart from terms which vanish by the equations of motion. Thus, in the continuum, it does not contribute to on-shell matrix elements for which the momentum inserted by the operator, Δp, vanishes. This is why the mixing of $O^{(\pm)}$ with O_p in the continuum is irrelevant to physical amplitudes. On the lattice, however, the breaking of chiral symmetry leads to $O(a)$ corrections to the PCAC equation even after renormalization [15]:

$$\langle h_1|\partial_\mu A_\mu - (m_s + m_d)P|h_2\rangle = \langle h_1|\bar{X}_A|h_2\rangle = O(a)\,. \tag{7}$$

Here $A_\mu = Z_A\bar{s}\gamma_\mu\gamma_5 d$ and $P = Z_P\bar{s}\gamma_5 d$ are the renormalized operators, m_d and m_s are renormalized quark masses, ∂_μ is a lattice derivative, and $h_{1,2}$ represent hadronic states. Thus the $K \to \pi\pi$ matrix element of the subtraction term is

$$\frac{C_p^{(\pm)}}{a}(m_c - m_u)(m_s - m_d)\langle\pi\pi|\bar{s}\gamma_5 d|K\rangle = \frac{C_p^{(\pm)}}{a}(m_c - m_u)\frac{m_s - m_d}{m_s + m_d}\frac{1}{Z_P}\langle\pi\pi|\partial_\mu A_\mu - \bar{X}_A|K\rangle\,. \tag{8}$$

The $\partial_\mu A_\mu$ term leads to the divergent contribution proportional to $\Delta p/a$. But when $\Delta p = 0$, the term proportional to $\bar{X}_A/a$ remains, and is of $O(1)$ up to logarithms. Thus the subtraction is still necessary, but the magnitude of the subtraction does not diverge in the continuum limit, and implies that it is sufficient to know $C_p^{(\pm)}$ with an error of $O(a)$.

In summary, to calculate the physical $K \to \pi\pi$ matrix elements of $O^{(\pm)}$ one needs both a model to do the analytic continuation from Euclidean to Minkowski space, and a method to subtract the operator $(m_c - m_u)O_p$ with appropriate coefficients $C_p^{(\pm)}$. Having summarised the problem, the remainder of the paper is devoted to illustrate several methods which can be used to extract the physical amplitudes.

3 Calculations with $\mathbf{m_s = m_d}$.

Bernard *et al.* proposed an ingenious solution to both problems [4]. They suggest working with $m_s = m_d$, and calculating the Euclidean amplitude in which all three particles are at rest:

$$\mathcal{A}^{(\pm)}_{\mathrm{m_s=m_d}} = \langle \pi(\vec{p}_1 = 0)\pi(\vec{p}_2 = 0)|O^{(\pm)}(\mu)|K(\vec{p}_K = 0)\rangle \Big|_{m_s=m_d} . \tag{9}$$

Setting $m_s = m_d$ causes O_p to vanish identically, and so removes the need for the subtraction. Working with the two pions at rest solves the problem of final state interactions. The final step in the method is to extrapolate from the unphysical amplitudes $\mathcal{A}^{(\pm)}_{\mathrm{dir}}$ to the physical amplitudes $\mathcal{A}^{(\pm)}_{\mathrm{phys}}$ using lowest order chiral perturbation theory. The result is that

$$\mathcal{A}^{(\pm)}_{\mathrm{phys}} = \frac{m^2_{K,\mathrm{phys}} - m^2_{\pi,\mathrm{phys}}}{2m^2_{K,\mathrm{lat}}} \, \mathcal{A}^{(\pm)}_{\mathrm{m_s=m_d}} \left[1 + O\left(\frac{m^2_K}{\Lambda^2_\chi} \right) \right] , \tag{10}$$

where $m_{K,\mathrm{lat}}$ is the mass of the lattice "kaon" used in calculating $\mathcal{A}^{(\pm)}_{\mathrm{m_s=m_d}}$. Due to the choice $m_s = m_d$, this is also the mass of the lattice "pion". $\Lambda_\chi \approx 4\pi f_\pi$ is the scale which determines the size of higher order terms in chiral perturbation theory. Such terms typically give 25% corrections if $m_{K,\mathrm{lat}} \approx m_{K,\mathrm{phys}}$. The direct method should thus be adequate to test the $\Delta I = 1/2$ rule at a semiquantitative level.

The main advantage of this method is its simplicity. It determines the coefficient of the contribution to $\mathcal{A}^{(\pm)}_{\mathrm{phys}}$ of leading order in chiral perturbation theory without the need for a subtraction. A further simplification is that one only needs to calculate the contractions (A) and (B) in fig. 1, because, due to CPS symmetry, the other two contractions vanish identically when $m_s = m_d$. In fact, since one can show that the physical amplitude depends on quark masses only at non-leading order in chiral perturbation theory, the contributions of diagrams (C) and (D), which are proportional to $m_s - m_d$, must be of non-leading order.

The main disadvantage of the method is its dependence on chiral perturbation theory. Such dependence is unavoidable once one has set $m_s = m_d$. Of particular concern is the fact that the final state interaction phase in the $I = 0$ two pion channel is substantial at the physical point $s = m^2_K$ (where s is the square of the two-pion centre-of-mass energy), even though it is a non-leading effect in chiral perturbation theory. Thus it is possible that the corrections in eq. (10) are larger than the estimate m^2_K/Λ^2_χ [13]. In principle one could reduce this error by using non-leading order chiral perturbation theory. This, however, requires knowledge of coefficients which are unavailable from experiment [16].

4 Alternative methods

We wish to develop methods which allow us to reduce, and ultimately remove, the dependence on chiral perturbation theory. To do this we must use quark masses which are closer to their physical values, and in particular we must consider $m_s > m_d$. Working with non-degenerate quarks implies that the mixing with O_p is present, and must either be subtracted, or shown to be unimportant. We have devised a number of new methods which accomplish this goal. In this section we describe, in order of increasing complexity, the three which require relatively small changes in methodology compared to the proposal of section 3. More speculative methods are described in subsequent sections.

Before describing our methods, we first comment on the the overall renormalization factors $Z^{(\pm)}(\mu a, g_0^2)$. These must be calculated for all the methods described in this section and in sections 3 and 5. In the past $Z^{(\pm)}$ have been determined using one-loop perturbation theory. Although this may be satisfactory for a test of the $\Delta I = 1/2$ rule, a non-perturbative determination is clearly preferable. We wish to point out that such a determination can be made by a straightforward extension of the non-perturbative methods used to normalize $\Delta S = 2$ operators using quark states [21]. The key observation is that the anomalous dimensions of $O^{(\pm)}$ are unchanged by mixing with the pseudoscalar density, and thus coincide with those of the operators

$$\bar{O}^{(\pm)} = \left(\bar{\psi}_1\gamma_\mu\psi_2\,\bar{\psi}_3\gamma_\mu\gamma_5\psi_4 + \bar{\psi}_1\gamma_\mu\gamma_5\psi_2\,\bar{\psi}_3\gamma_\mu\psi_4\right) \pm \left(2 \leftrightarrow 4\right), \tag{11}$$

where the subscripts $1, 2, 3, 4$ label four different quark flavours. Only open current-current diagrams need be calculated; penguin-type diagrams with quark loops do not contribute. The non-perturbative calculation of the renormalized $\bar{O}^{(\pm)}$ is currently underway [18].

4.1 Method 1

We first describe the method and then explain why it is valid. The ingredients are

- Work with a non-perturbatively $O(a)$ improved fermion action, for which there are no errors of $O(a)$ in the spectrum, and the on-shell amplitudes of the improved currents obey the continuum chiral Ward identities up to $O(a^2)$. This can be accomplished using the methods of ref. [17].
- Use the lattice four-fermion operator, suitably normalized, but do not subtract the O_p term. In other words, set $C_p^{(\pm)} = 0$.
- Choose quark masses such that $m_K = 2m_\pi$.
- Calculate the $K \to \pi\pi$ amplitudes with all particles at rest. Given the choice of meson masses this means that $\Delta p = 0$. The resulting amplitudes are denoted $\mathcal{A}^{(\pm)}_{m_K=2m_\pi}$.
- Determine the physical amplitudes using lowest order chiral perturbation theory, which gives

$$\mathcal{A}^{(\pm)}_{\rm phys} = \frac{m^2_{K,\rm phys} - m^2_{\pi,\rm phys}}{m^2_{K,\rm lat} - m^2_{\pi,\rm lat}}\,\mathcal{A}^{(\pm)}_{m_K=2m_\pi}\left[1 + O\left(\frac{m_K^2}{\Lambda_\chi^2}\right)\right]. \tag{12}$$

The validity of this method is based on the following observations. If one uses fully $O(a)$ improved fermions, and the fully $O(a)$ improved axial current, A^I_μ, and pseudoscalar density, P^I, then the corrections to the PCAC equation (7) are of $O(a^2)$ rather than of $O(a)$ [17]. Furthermore, the improved pseudoscalar density is simply proportional to the bare lattice operator $P^I = Z_P^I \bar{s}\gamma_5 d$. Thus the argument following eq. (7) now implies that the matrix element of the subtraction term is

$$\frac{C_p^{(\pm)}}{a}(m_c - m_u)(m_s - m_d)\langle\pi\pi|\bar{s}\gamma_5 d|K\rangle = \frac{C_p^{(\pm)}}{a}(m_c - m_u)\frac{m_s - m_d}{m_s + m_d}\frac{1}{Z_P^I}\langle\pi\pi|\partial_\mu A^I_\mu|K\rangle + O(a)\,. \tag{13}$$

The important point is that the discretization error on the r.h.s. of this result is now of $O(a)$ rather than of $O(1)$ [cf. eq. (8)]. This means that if we can set $\Delta p = 0$, so that the term

involving $\partial_\mu A_\mu^I$ vanishes, the subtraction leads to corrections only of $O(a)$. Since these are of the size that we are neglecting, it is not necessary to do the subtraction. Note that to make this argument, we do not need to know the form of A_μ^I, nor the value of the renormalization constants Z_A^I and Z_P^I.

Since we have chosen quark masses such that $m_K = 2m_\pi$, and chosen all particles to have $\vec{p} = 0$, we do have $\Delta p = 0$. Furthermore, the use of pions at rest allows us to avoid final state interactions, as in the $m_s = m_d$ method.

We view this new method as complementary to the method of section 3. It is only slightly more difficult to implement practically. One complication is the need to use fully $O(a)$ improved Wilson fermions. This is, however, becoming standard in numerical simulations, now that the necessary "clover" coefficient c_{SW} has been determined non-perturbatively for quenched QCD [17]. Note that the method of section 3 can be used equally well with improved fermions, but that this does not provide any particular advantage over unimproved fermions. The results for $K \to \pi\pi$ amplitudes will have errors of $O(a)$ in both cases. To remove these errors would require not only improving the action but also improving the operators $O^{(\pm)}$.

A second potential technical complication is the need to include all four Wick contractions of fig. 1. One can, however, consistently drop diagrams (C) and (D) since they are non-leading in chiral perturbation theory, while we are relying in eq. (12) on the leading order term.

A possible advantage of this method is that it requires a smaller extrapolation to the physical point, since one is using quark masses which are closer to those of the physical quarks. Furthermore, if one includes diagrams (C) and (D), one can perform a partial test of the convergence of the chiral expansion. Diagram (C) is straightforward to calculate, while the disconnected diagram (D) presents more technical difficulties. We expect, however, that since (D) is Zweig forbidden, its contribution will be smaller than that of (C).

4.2 Method 2

A potential problem with the previous method is that the momentum inserted will not, in practice, be exactly zero. If so, the part of the O_p term proportional to $\partial_\mu A_\mu^I$ will give a "small" but divergent contribution proportional to $\Delta E/a$, where $\Delta E = m_K - 2m_\pi$. We can mitigate this potential problem by determining $C_p^{(\pm)}$ non-perturbatively. We propose to do so by applying the condition

$$\langle 0|O^{(\pm)}(\mu)|K\rangle = 0\,. \tag{14}$$

We then proceed as in the previous method, except now keeping the subtraction term. Note that implementation of this condition does not require knowledge of the overall normalizations $Z^{(\pm)}$.

For this relatively small increase in effort we effectively determine the $O(1)$ part of $C_p^{(\pm)}$, and thus reduce the error from the incomplete subtraction of the O_p term by one power of a. In other words, whereas in Method 1 ignoring the subtraction term leads to an error $\sim \Delta E/a + O(a)$, in Method 2 the residue after implementing eq. (14) is $\sim \Delta E + O(a^2)$. Since there are other sources of $O(a)$ errors, the total error in the $K \to \pi\pi$ matrix element is $\sim \Delta E + O(a)$. Thus the extrapolation to $\Delta E = 0$ will be considerably less delicate. Furthermore, since the method no longer relies on the improved PCAC relation, it can be used with unimproved, or tree-level improved, Wilson fermions.

We now explain how the condition (14) is justified. At leading order in chiral perturbation theory, the $K \to$vacuum matrix element takes the form [14] ($f_\pi \sim 132$ MeV)

$$\langle 0|O^{(\pm)}(\mu)|K^0\rangle = i\,\delta^{(\pm)}\,\frac{m_K^2 - m_\pi^2}{f_\pi}\,. \tag{15}$$

The coefficients $\delta^{(\pm)}$ do not contribute to the physical $K \to \pi\pi$ matrix element (see eq. (17) below), and thus can be set to zero. A similar unphysical arbitrariness remains away from the chiral limit. The point is that we can always redefine the renormalized operators as follows [19]

$$O^{(\pm)} \longrightarrow O^{(\pm)} + F^{(\pm)}(m_c - m_u)O_p\,. \tag{16}$$

where the finite coefficients $F^{(\pm)}$ are of $O(\Lambda_{\rm QCD})$ but otherwise arbitrary. This redefinition does not change the physical matrix elements of $O^{(\pm)}$ since O_p is a total divergence. But it does affect matrix elements in which momentum is inserted, such as those appearing in the condition of eq. (14). This condition can thus be fulfilled by an appropriate choice of $F^{(\pm)}$. In particular, it can also be used if the strange quark is replaced by the bottom quark.

The possibility of redefining the renormalized operators by finite terms, as in eq. (16), implies that dependence of the $K \to \pi\pi$ matrix elements on ΔE is of $O(1)$ and does not vanish with the lattice spacing as one would naïvely expect.

4.3 Method 3

The previous methods require the use of quark masses whose ratio m_s/m_d differs from its physical value. To move closer to physical quark masses while holding $\Delta E = 0$ we must work with final states in which $\vec{p}_\pi \neq 0$. This forces us to deal with the Maiani-Testa theorem. We propose doing so using the method of Ciuchini *et al.* (CFMS) [20]. Given an analytic parameterization of the two pion scattering amplitude (e.g. resonance dominance) CFMS show how, in principle, one can extract the magnitude and phase of the physical amplitude by studying the Euclidean amplitude as a function of time for a variety of final pion momenta.

An assumption made by CFMS is the "smoothness" of the off-shell amplitudes. In the absence of resonances coupled to the final-state mesons, this requires that the $K \to \pi\pi$ amplitudes do not vary rapidly with Δp. If there is a nearby resonance, CFMS assume that it dominates the momentum dependence of the amplitudes. The "smoothness" hypothesis is, in this case, the assumption that the couplings of the resonance to the final-state particles are smooth functions of the external momenta. It is then is possible to find a simple parametrization of the amplitudes which describes their rapid variation (due to the presence of the resonance) with Δp.

Thus we propose using the method of CFMS after fixing $C_p^{(\pm)}$ with the conditions eq. (14). This removes the terms proportional to $\Delta p/a$ which might violate the smoothness hypothesis. Since this method makes no use of chiral perturbation theory, it can be applied also B decays. The only restriction is that the GIM mechanism must be operative, which limits one to processes which do not involve top quark loops.

5 K → π Matrix Elements

In this section we reconsider the method suggested in ref. [3] which uses the $K \to \pi$ matrix elements of the positive parity part of the weak Hamiltonian. This method relies on chiral perturbation theory, and in this respect we expect it to be of comparable accuracy to the direct method of Bernard *et al.* and the first two of new methods introduced in the previous section. Since only single-particle states are involved, the advantage of the "$K \to \pi$ method" is that it is technically easier to extract the relevant matrix elements. The disadvantage is that the operator mixing problem is much more complicated than for the negative parity part of $\mathcal{H}_{\rm eff}^{W}$, making an accurate evaluation of the matrix element of the renormalized operator difficult.

We begin by recalling how the physical amplitude is obtained from the knowledge of the properties of the $K \to \pi$ amplitude exploiting the Soft-Pion Theorems (SPTs). At leading order in chiral perturbation theory the physical amplitude takes the form (for $\Delta p = 0$)

$$\langle \pi^+\pi^-|O^{(\pm)}(\mu)|K^0\rangle = i\,\gamma^{(\pm)}\,\frac{m_K^2 - m_\pi^2}{f_\pi}\,. \tag{17}$$

The coefficients $\gamma^{(\pm)}$ appear also in the expression for the $K \to \pi$ matrix element

$$\langle \pi^+(p)|O^{(\pm)}(\mu)|K^+(q)\rangle = -\delta^{(\pm)}\,\frac{m_K^2}{f_\pi^2} + \gamma^{(\pm)}\,p\cdot q\,. \tag{18}$$

By studying this matrix element as a function of $p \cdot q$ one can, in principle, determine $\gamma^{(\pm)}$, from which we obtain the $K \to \pi\pi$ matrix elements up to corrections of order m_K^2/Λ_χ^2.

The expression for the positive-parity components of the renormalized $O^{(\pm)}$ is [3] [1]

$$\begin{aligned} O^{(\pm)}(\mu) &\equiv Z^{(\pm)}(\mu a, g_0^2)\, O^{(\pm)}_{\rm sub} \\ &= Z^{(\pm)}(\mu a, g_0^2)\left[O^{(\pm)}(a) + \sum_{i=1}^{4} C_i^{(\pm)} O_i^{(\pm)}(a)\right. \\ &\quad \left. + (m_c - m_u) C_\sigma^{(\pm)} O_\sigma(a) + (m_c - m_u)\frac{C_s^{(\pm)}}{a^2} O_s(a)\right]\,. \end{aligned} \tag{19}$$

Here $O_i^{(\pm)}$ are four-fermion operators of dimension 6 which have different chirality from $O^{(\pm)}$. They are listed in refs. [11], [21]–[23]. The remaining operators are of lower dimension

$$O_\sigma = g_0 \bar{s}\sigma_{\mu\nu}G_{\mu\nu}d\,, \tag{20}$$

$$O_s = \bar{s}d\,. \tag{21}$$

Comparing the result in eq. (19) with that for the negative parity parts of $O^{(\pm)}$, eq. (6), we see that CPS symmetry provides much weaker constraints for the parity conserving parts. In particular, the loss of the factor of $m_s - m_d$ means that the mixing with the scalar density diverges like $1/a^2$.

[1] Although, for convenience, we use the same symbol, $Z^{(\pm)}$, the values of the overall renormalization constant for the positive parity part is in general different from that for the parity violating part, as a result of the explicit chiral symmetry breaking on the lattice. The operators $O^{(\pm)}$ in this section also refer to their positive parity components.

Reference [3] suggested the following approach for determining the mixing coefficients: use perturbative values for the coefficients of the dimension 6 operators and O_σ, but determine $C_s^{(\pm)}$ non-perturbatively. The latter determination is to be made by adjusting $C_s^{(\pm)}$ until the momentum independent part of the $K \to \pi$ matrix element in eq. (18) vanishes, i.e. $\delta^{(\pm)} = 0$. This is the positive parity analogue of the condition (14), and can be justified by similar arguments.

One problem with this approach is that a perturbative determination of mixing coefficients is often not reliable. This is exemplified by the case of $\Delta S = 2$ operators, where the operator with non-perturbatively determined coefficients has, to a very good approximation, the expected chiral behaviour, while that with perturbative coefficients does not [21, 23]. Nevertheless, this approach should be pursued, because it could provide semiquantitative results for the $\Delta I = 1/2$ rule. To date, no useful results have been obtained.

There are three approaches which have been proposed to determine the subtraction coefficients of four-fermion operators in a non-perturbative way: Gauge Invariant Ward Identities (GIWIs) [3], Ward Identities on quark states [23] and non-perturbative renormalization between quark states at large external momenta [21].

We start with a discussion of the GIWIs, reformulated in the light of the recent progress in the exploitation of Ward identities in the context of improved actions [17]. As we shall explain below, this is the most promising approach in the presence of mixing with lower dimensional operators (and is more general than the present case). Consider the following Ward identity in the chiral limit:

$$\sum_{y\in\mathcal{R}} \langle \partial_\mu A_\mu^f(y)\, O_{\rm sub}^{(\pm)}(0)\, \Phi(x_1, x_2, \cdots, x_n)\rangle = -i\langle \frac{\delta O_{\rm sub}^{(\pm)}(0)}{\delta\alpha^f}\, \Phi(x_1, x_2, \cdots, x_n)\rangle \ , \qquad (22)$$

where $\mathcal{R}$ is a region of space-time containing the origin bounded by two hyperplanes $y_4 = -t_a$ and $y_4 = t_b$ and f labels the flavour component of the axial transformation. Φ represents a multilocal gauge-invariant operator, with $x_1, x_2, \cdots, x_n$ all lying outside $\mathcal{R}$. $\delta O_{\rm sub}^{(\pm)}/\delta\alpha^f$ denotes the variation of the operators under infinitesimal axial rotations of the fields. As shown in ref. [3], in the chiral limit, there is a unique choice of the coefficients of the operators which belong to different chiral representations, i.e. $C_i^{(\pm)}$, $C_\sigma^{(\pm)}$, $C_s^{(\pm)}$ and $C_p^{(\pm)}$, such that the subtracted operators $O_{\rm sub}^{(\pm)}$ satisfy this Ward identity. By varying the points $x_1, x_2, \cdots, x_n$, eq. (22) corresponds to an overdetermined set of linear inhomogeneous equations which, in principle, allow for the determination of all the mixing coefficients. Using the property that the axial current is conserved in the chiral limit, it is convenient to rewrite eq. (22) as follows:

$$\sum_{\vec{y}} \langle \Big(A_4^f(\vec{y}, t_b) - A_4^f(\vec{y}, -t_a) \Big)\, O_{\rm sub}^{(\pm)}(0)\, \Phi(x_1, x_2, \cdots, x_n)\rangle = -i\langle \frac{\delta O_{\rm sub}^{(\pm)}(0)}{\delta\alpha^f}\, \Phi(x_1, x_2, \cdots, x_n)\rangle \ . \qquad (23)$$

The above equation shows that there are no contact terms arising upon integration over y [17]. The absence of contact terms implies that we do not have to include the mixing with operators that vanish by the equations of motion. Although these operators do not contribute to physical matrix elements, in general (see below) they must be taken into account in the determination of the mixing coefficients. Notice that the coefficients determined in the chiral limit are sufficient to predict unambiguously the physical $K \to \pi\pi$ amplitudes.

The other two non-perturbative methods mentioned above use quark and gluon correlators in a fixed gauge, and either impose normalization conditions at large Euclidean momenta

[21], or enforce the Ward Identities on quark Green functions [23]. The problem with these methods is that they require the inclusion of two additional classes of operator[2]:

1. Gauge invariant operators which vanish by the equation of motion. These do not contribute to on-shell matrix elements, but do contribute to the off-shell correlators used in these methods. In the present case there are three such operators of low enough dimension

$$\begin{aligned} &\bar{s}(\overrightarrow{\not{D}} + m_d)d + \bar{s}(-\overleftarrow{\not{D}} + m_s)d\,, \\ &\bar{s}(\overrightarrow{\not{D}} + m_d)^2 d + \bar{s}(-\overleftarrow{\not{D}} + m_s)^2 d\,, \\ &\bar{s}(-\overleftarrow{\not{D}} + m_s)(\overrightarrow{\not{D}} + m_d)d\,. \end{aligned} \tag{24}$$

   The coefficient of the first is proportional to $(m_c - m_u)/a$, while that of the latter two are proportional to $(m_c - m_u)$. These operators also appear in the continuum, but they can be removed in perturbation theory by studying how the form factors behave as one goes on-shell. This is not possible in a numerical simulation.

2. Operators which are not gauge invariant. These are, however, constrained to be either BRST invariant, or to vanish by the equations of motion [25]. Note that there is an exact BRST symmetry on the lattice after gauge fixing [26]. There are two such operators of low enough dimension, and with positive CPS parity,

$$(m_c - m_u)\left[\bar{s}\overleftarrow{\partial}(\overrightarrow{\not{D}} + m_d)d - \bar{s}(-\overleftarrow{\not{D}} + m_s)\overrightarrow{\partial} d\right]\,, \tag{25}$$

$$(m_c - m_u)\left[\bar{s}\overrightarrow{\partial}(\overrightarrow{\not{D}} + m_d)d - \bar{s}(-\overleftarrow{\not{D}} + m_s)\overleftarrow{\partial} d\right]\,. \tag{26}$$

By varying the external momenta, and using suitable projectors [21, 23], it may be possible, in principle, to separate the contributions of these operators. The inclusion of five additional operators, when the normalization conditions are imposed on quark states in momentum space, seems, however, to make the possibility of a non-perturbative determination of the mixing coefficients quite remote in practice.

These problems can be avoided, by working in configuration space, rather than in momentum space. This can be achieved by studying (unamputated) quark Green functions of the form

$$\langle \psi_1(x_1)\psi_2(x_2)O^{(\pm)}(0)\bar{\psi}_3(x_3)\bar{\psi}_4(x_4)\rangle\,, \tag{27}$$

where all the points are separated. In this case none of the operators which vanish by the equation of motion can appear. To enforce the Ward identities, one uses eq. (23) with a non-gauge invariant $\Phi(x_1, \cdots, x_4) = \psi_1(x_1)\psi_2(x_2)\bar{\psi}_3(x_3)\bar{\psi}_4(x_4)$. The expectation value has to be evaluated in a fixed gauge. Although, in this case, we have the same number of coefficients to determine, we prefer the GIWI method, since it requires the evaluation of gauge invariant correlation functions only.

6 Construction of $\mathcal{H}_{\rm eff}^{\rm W}$ from first principles

In this section we propose a method which, in principle, avoids the difficulties caused by mixing with lower dimension operators, and which automatically gives the effective weak

[2] For a more detailed discussion of why such operators appear see ref. [24].

Hamiltonian with the correct normalization. In addition, it allows one to construct an improved weak Hamiltonian, i.e. one having errors of $O(a^2)$, given only the improved versions of the weak currents. The method does not use chiral perturbation theory, and thus applies equally well to the $\Delta S = 1$, $\Delta C = 1$ and $\Delta B = 1$ parts of the weak Hamiltonian. The method is speculative in the sense that it is likely to require more computational power than is presently available, although we expect it to be practical with the advent of Teraflops machines.

The standard construction of the non-leptonic weak Hamiltonian begins with the expression

$$\mathcal{H}_{\rm eff}^W = g_W^2 \int d^4x\, D_{\rho\nu}^W(x; M_{_W}) T\left[J_{\rho L}(x) J_{\nu L}^\dagger(0)\right] \ , \tag{28}$$

where

$$D_{\rho\nu}^W(x; M_{_W}) = \int d^4p \, \frac{{\rm e}^{ipx}}{p^2 + M_{_W}^2} (\delta^{\rho\nu} - \frac{p^\rho p^\nu}{M_{_W}^2}) \tag{29}$$

is the W-boson propagator and $J_{\rho L}$ is the (left-handed) hadronic weak current. One then performs an operator product expansion (OPE) on the product of the two currents in eq. (28), which is justified by the observation that the dominant contribution to the integral comes from distances $|x| \ll M_{_W}^{-1}$. For physical amplitudes, one obtains in this way

$$\langle h|\mathcal{H}_{\rm eff}^W|h'\rangle = \frac{G_F}{\sqrt{2}} \sum_i C_i(\mu, M_{_W}) M_{_W}^{6-d_i} \langle h|O^{(i)}(\mu)|h'\rangle \ , \tag{30}$$

where d_i is the dimension of the operator $O^{(i)}(\mu)$, and functions $C_i(\mu, M_{_W})$ result from the integration of the Wilson expansion coefficients, $c_i(x;\mu)$ (defined in eq. (33) below), with the W-propagator. Schematically, suppressing Lorentz indices, one has

$$C_i(\mu, M_{_W}) M_{_W}^{6-d_i} = \int d^4x\, D^W(x; M_{_W}) c_i(x;\mu) \ . \tag{31}$$

The $O^{(i)}(\mu)$ are quark and/or gluon operators renormalized at the subtraction point μ. The functions $C_i(\mu, M_{_W})$ are evaluated in perturbation theory and their running with μ is dictated by the renormalization group equation which follows from the μ-independence of the l.h.s. of eq. (30).

The sum in the expansion (30) is over operators of increasing dimension. We consider in the following only operators with dimensions $d_i \leq 6$, since the contribution from operators with $d_i > 6$ is suppressed by powers of $1/M_{_W}$.

All the intricacies of operator mixing in the definition of the finite and renormalized operators, $O^{(i)}(\mu)$, come about because the integrals in (28) and (31) are extended down to the region of extremely small x. The complicated mixing for the $O^{(i)}(\mu)$'s in terms of bare operators arises from contact terms when the separation of the two currents goes to zero (i.e. when $|x|$ is of the order of a). The problem is particularly bad because chiral symmetry is broken by the lattice regularization. This observation suggests that a simple way to avoid these complications is to implicitly define the renormalized operators by enforcing the OPE on the lattice for distances $|x|$ much larger than the lattice spacing a. We imagine proceeding in the following way:

1. Take the T-product of two properly normalized weak currents, $J_{\rho L}(x) J_{\rho L}^\dagger(0)$. If required these currents can be improved.

2. Measure the hadronic matrix element $\langle h|T[J_{\rho L}(x)J^\dagger_{\rho L}(0)]|h'\rangle$ in a Monte Carlo simulation, as a function of x for $|x| \to 0$ in the region

$$a \ll |x| \ll \Lambda^{-1}_{QCD}\,. \tag{32}$$

3. Extract the numbers $\langle h|O^{(i)}(\mu)|h'\rangle$ by fitting in the region (32) the x-behaviour of $\langle h|T[J_{\rho L}(x)J^\dagger_{\rho L}(0)]|h'\rangle$ to the formula

$$\langle h|T\left[J_{\rho L}(x)J^\dagger_{\rho L}(0)\right]|h'\rangle = \sum_i c_i(x;\mu)\langle h|O^{(i)}(\mu)|h'\rangle\,, \tag{33}$$

where the Wilson coefficients $c_i(x;\mu)$ are determined by continuum perturbation theory using any standard renormalization scheme. The scale μ should be chosen so that $1/\mu$ lies in the range defined by eq. (32). Since we only consider operators of dimension 6 or lower, the T-product differs from the right-hand side of eq. (33) by terms of $O(|x|^2\Lambda^2_{\rm QCD})$, which is an estimate of the size of the systematic errors in this procedure. Note that in eq. (33) an average over the points x and $-x$ (schematically $J(x)J(0) \to 1/2(J(x)J(0)+J(-x)J(0))$) is implied in order to eliminate from the OPE terms which do not appear in physical amplitudes because of the integration over x in eq. (28). These terms, however, would appear on the r.h.s. of eq. (33) were we to not perform this average.

4. Insert the numbers $\langle h|O^{(i)}(\mu)|h'\rangle$ determined in this way into the expression for the matrix elements of $\mathcal{H}^W_{\rm eff}$, finally obtaining eq. (30).

For the implementation of this procedure, what is required is the existence of a window, eq. (32), in which the distance between the two currents is small enough so that perturbation theory can be used to determine the expected form of the OPE, but large enough that lattice artifacts are small. These artifacts will be suppressed by powers of a/x. Clearly the existence of such a window requires that we have a sufficiently small lattice spacing. At the same time the physical volume of the lattice must be sufficiently large to allow the formation of hadrons.

A few remarks may be useful at this point:

- The method determines directly the "physical" matrix elements of the operators appearing in the OPE of the two currents, i.e. the matrix elements of the finite, renormalized operators $O^{(i)}(\mu)$, without any reference to the magnitude of the W-mass. Thus we do not need to probe distances of $O(1/M_{_W})$ with lattice calculations.

- If the action and currents are improved, then the resulting matrix elements of $O^{(i)}(\mu)$, and thus of $\mathcal{H}^W_{\rm eff}$, will also be improved.

- The μ-dependence of the matrix elements of the operators $O^{(i)}(\mu)$ is given trivially by that of the (perturbative) Wilson coefficients, $c_i(x;\mu)$. It compensates the related μ-dependence of the functions $C_i(\mu, M_{_W})$ in such a way that the l.h.s of eq. (30) is independent of the choice of the subtraction point. A similar comment holds for the dependence on renormalization scheme.

- Unlike the methods discussed in previous sections, this approach automatically yields hadronic amplitudes that are properly normalized (in the renormalization scheme in which the Wilson coefficients appearing in eq. (33) are computed).

We now discuss the feasibility of the method in more detail. The critical step is fitting to the form predicted by the OPE, eq. (33). Typically more than one operator contributes to the sum, so one must be able to separate the contributions using their different dependence on x. The operators of interest are of dimension 6, and thus have Wilson coefficients which vary logarithmically with x. At leading order the form is

$$c_i(x;\mu) \propto \left(\frac{\alpha_s(1/x)}{\alpha_s(\mu)}\right)^{\frac{\gamma_0^{(i)}}{2\beta_0}} = 1 + \frac{\alpha_s}{4\pi}\gamma_0^{(i)} \log(x\mu) + \dots \,, \tag{34}$$

where $\gamma_0^{(i)}$ is the one-loop anomalous dimension of the operator $O^{(i)}$, and β_0 is coefficient of the one-loop term in the β-function. By contrast, the coefficients of lower dimension operators diverge as powers of $1/x$ (up to logarithmic corrections). Thus if lower dimension operators are present they will dominate at short distances, and it will be very difficult to pick out the matrix elements of the dimension 6 operators. If, on the other hand, only dimension 6 operators appear then it may be possible to separately determine their matrix elements. How feasible this is depends on how large a range of x we can use, and on the magnitude of the differences between the anomalous dimensions.

Fortunately, in the cases of interest, there are no operators of dimension lower than 6. Consider, for example, the $\Delta S = 1$ part of $\mathcal{H}_{\rm eff}^W$. The operators which can appear in the OPE are $O^{(\pm)}$ [defined in eq. (3)], and in addition

$$O' = (m_c^2 - m_u^2)\, \bar{s}(\overrightarrow{D_\mu} - \overleftarrow{D_\mu})\gamma_\mu^L d \,. \tag{35}$$

The GIM mechanism requires O' to vanish when $m_c = m_u$, while chiral symmetry requires both the quarks to be left-handed and that the GIM factor be quadratic in the quark masses. Although this operator looks new, its negative parity part is, by the equations of motion, proportional to $(m_c^2 - m_u^2)\, O_p$, where O_p is defined in eq. (4), and its positive parity part is proportional to $(m_c^2 - m_u^2)\,(m_s + m_d)\, O_s$. So these are the same operators we encountered in sections 2 and 5, except for the overall factors. Since O' has dimension 6,, its coefficient function depends only logarithmically on x.

To determine the matrix elements using eq. (33) we need the anomalous dimensions, which for the three operators are

$$\gamma_0^{(+)} = 4, \qquad \gamma_0^{(-)} = -8, \qquad \gamma_0' = 16 \ . \tag{36}$$

In fact, the contribution of O' to the r.h.s. of eq. (33) can be determined separately since its matrix element does not require any subtraction and can be calculated directly. As for $O^{\pm}$, since their anomalous dimensions are well separated from one another, it may be possible to extract the corresponding matrix elements and then construct the physical amplitude of $\mathcal{H}_{\rm eff}^{\Delta S=1}$.

An important element of the procedure proposed in this section is that since it is the continuum OPE which determines the operators which appear, these are restricted by continuum symmetries. This is because, for $|x| \gg a$, the lattice OPE matches that of the continuum with discretization errors suppressed by powers of a/x.

The previous discussion shows how we can, in principle, remove the contribution of the scalar and pseudoscalar densities from any hadronic matrix element. Not only does the method work for both positive and negative parity parts of $\mathcal{H}_{\rm eff}^W$, but it also works if the

weak Hamiltonian carries momentum Δp. Thus the simplest way to test the method may be to calculate $K \to \pi$ matrix elements and then use chiral perturbation theory to relate these to the physical amplitude, as described in the previous section. The problems of operator mixing described in sec. 5 do not apply to the new method.

Of course, for physical matrix elements one does not need to worry about the subtraction of O'. This is because the matrix elements determined by this method are those of the continuum operator, up to discretization errors.

We end this section with an observation on the computational feasibility of this approach. The main difficulty is to have a sufficiently large range of values of $|x|$ in order to separate the contributions from the different operators, and yet to satisfy the condition (32). These constraints make the method difficult at present, and it will only be fully exploited when Teraflops machines become available.

7 A propagating top quark

For CP violating processes in kaon decays, or for B decays where top-penguin diagrams enter at a Cabibbo-allowed level, the strategies described in sec. 4 and 6 for the negative parity operators fail, because the GIM mechanism is not operative. In particular $O^{(\pm)}$ mix with all the penguin operators (see below). This makes the calculation of the mixing matrix of comparable difficulty to that for the positive parity operators described in sec. 5. For positive parity operators, the analysis of section 5 still applies. The difference is that the mixing coefficients of the magnetic operator and scalar density become more divergent [3] and this can make the numerical determination of the renormalized operators less precise. In order to circumvent these problems we propose two methods involving a fictitious top quark (with mass $\widetilde{m}_t$) which is light enough to propagate on the lattice.

The basic idea is to work with two different scales: the first, μ, is larger than $\widetilde{m}_t$, so that the corresponding operator basis is as in the previous sections ($O^{(\pm)}$); the second, μ', is smaller than $\widetilde{m}_t$ so that a full set of penguin operators is generated. The matrix elements of the operators renormalized at the scale μ are computed numerically following the strategies explained in secs. 4–6. By matching the result to the amplitude expressed in terms of operators renormalized at μ', we extract their matrix elements. In this way, at least in principle, we can obtain the matrix elements of the penguin operators without directly computing them. We now present the details of this procedure.

At scales μ' below m_t, when GIM is not operative, the form of the $\Delta S = 1$ effective Hamiltonian is

$$\begin{aligned} \mathcal{H}_{eff}^{\Delta S=1} &= \frac{G_F}{\sqrt{2}}\Big[\lambda_u \Big(C_1(\mu', M_{_W})\,(Q_1^u(\mu') - Q_1(\mu')) + C_2(\mu', M_{_W})\,(Q_2^u(\mu') - Q_2(\mu'))\Big) \\ &- \lambda_t\, \vec{C}(\mu', M_{_W}, m_t) \cdot \vec{Q}(\mu')\Big] \end{aligned} \tag{37}$$

where $\lambda_q = V_{qd}V_{qs}^*$, $q = u, c, t$ (λ_c is eliminated by using the unitarity relation $\lambda_c = -\lambda_u - \lambda_t$). Here $\vec{Q}$ contains the "penguin" operators

$$\vec{Q}(\mu') \equiv (Q_1(\mu'), Q_2(\mu'), \ldots, Q_6(\mu')) \tag{38}$$

and $\vec{C}$ are the corresponding coefficients

$$\vec{C}(\mu') \equiv (C_1(\mu', M_{_W}), C_2(\mu', M_{_W}), C_3(\mu', M_{_W}, m_t) \ldots, C_6(\mu', M_{_W}, m_t))\,. \tag{39}$$

A convenient basis of operators when QCD corrections are taken into account is [33]–[36]

$$\begin{aligned} Q_1 &= (\bar{s}d)_{(V-A)}(\bar{c}c)_{(V-A)}\,, \\ Q_2 &= (\bar{s}c)_{(V-A)}(\bar{c}d)_{(V-A)}\,, \\ Q_{3,5} &= (\bar{s}d)_{(V-A)}\sum_q(\bar{q}q)_{(V\mp A)} \\ Q_4 &= \sum_q(\bar{s}q)_{(V-A)}(\bar{q}d)_{(V-A)} \\ Q_6 &= -2\sum_q(\bar{s}q)_{(S+P)}(\bar{q}d)_{(S-P)}\,. \end{aligned} \tag{40}$$

$Q_1^{u,t}$ and $Q_2^{u,t}$ are the analogous operators to Q_1 and Q_2 with the up- and top-quark replacing the charmed one. Here the subscripts $(V \pm A)$ and $(S \pm P)$ indicate the chiral structures, and the sum over quarks q runs over the active flavours at the scale μ'. For simplicity we ignore electroweak penguin and magnetic operators; it is straightforward to generalize the following discussion to include them. The coefficient functions appearing above have been calculated up to non-leading order in perturbation theory in refs. [27]–[32]. The effective Hamiltonian relevant for $\Delta B = 1$ decays is simply obtained by replacing the s quark with the b quark.

The part of $\mathcal{H}_{\rm eff}^{\Delta S=1}$ proportional to λ_u is the same as that considered above in eq. (2), and has been the focus of discussion for much of the paper. We have simply re-expressed it here in the new operator basis, in terms of which

$$O^{\pm} = (Q_1^u - Q_1) \pm (Q_2^u - Q_2)\,, \tag{41}$$

$$C_{\pm}(\mu, M_{_W}) = \frac{1}{2}\left[C_1(\mu, M_{_W}) \pm C_2(\mu, M_{_W})\right]\,. \tag{42}$$

This part of $\mathcal{H}_{\rm eff}^{\Delta S=1}$ gives the dominant contribution to CP conserving $K \to \pi\pi$ amplitudes. These amplitudes can be calculated using the methods presented in secs. 4–6.

The difficulties arise for the part of $\mathcal{H}_{\rm eff}^{\Delta S=1}$ proportional to λ_t, which gives rise to CP violation in kaon decays. This part contains the penguin operators Q_i, whose matrix elements are not protected by the GIM mechanism. To write a renormalized version of these operators requires subtracting O_p and $\widetilde{O}_\sigma$ with appropriate coefficients, and accounting for mixing with all the other operators Q_j. This is true for both positive and negative parity sectors. The methods of this section are designed to avoid these difficulties.

We do so by introducing a dynamical top quark so as to keep the GIM mechanism operative. This will, however, be a fictitious top quark with mass satisfying

$$1/a \gg \widetilde{m}_t \gg m_c \gg \Lambda_{\rm QCD}\,. \tag{43}$$

In other words, our top is light enough to propagate on the lattice, but, like the physical top, it is heavier than the charm quark. As explained below, by using the fictitious top we can extract the matrix elements $\langle h|Q_i(\mu')|h'\rangle$ which can then be inserted into the expression for $\mathcal{H}_{\rm eff}^{\Delta S=1}$, eq. (37). In this respect the method is similar to that of sec. 6. For purposes of illustration we will restrict the discussion below to negative parity operators.

We begin with the two matrix elements

$$\mathcal{M}_i(\mu, \widetilde{m}_t) \equiv \langle h|Q_i(\mu) - Q_i^t(\mu)|h'\rangle\,, \quad i = 1, 2 \tag{44}$$

evaluated at a renormalization scale satisfying $a^{-1} \sim \mu \gg \widetilde{m}_t$. Since GIM is operative, the analysis of sec. 2 applies (with $m_u \to m_t$). Thus we can define $Q_i(\mu) - Q_i^t(\mu)$ in terms of bare lattice operators by

$$Q_i(\mu) - Q_i^t(\mu) = Z_{ij}(\mu a, g_0^2) \left[Q_j(a) - Q_j^t(a) + (m_c - \widetilde{m}_t) \frac{C_p^{(j)}}{a} O_p(a) \right] , \tag{45}$$

where $i, j = 1, 2$ and the subtraction coefficients $C_p^{(j)}$ are determined by enforcing

$$\langle 0 | Q_j(a) - Q_j^t(a) + (m_c - \widetilde{m}_t) \frac{C_p^{(j)}}{a} O_p(a) | K \rangle = 0 \, . \tag{46}$$

The Z_{ij} are related by a simple change of basis to the $Z^{(\pm)}$ of eq. (6), and can be calculated either perturbatively or non-perturbatively. In this way we can obtain $\mathcal{M}_i(\mu, \widetilde{m}_t)$ from a lattice calculation, as a function of $\widetilde{m}_t$, for some choice of μ. A simple choice is $\mu \approx 1/a$.

On the other hand, we can also consider the same matrix elements for a renormalization scale $\widetilde{m}_t \gg \mu' \gg m_c$. In this case, the GIM mechanism is not operative, and the matrix elements can be expressed in terms of the six operators which appear in eq. (40)

$$\mathcal{M}_i(\mu, \widetilde{m}_t) = \sum_{j=1,6} \widehat{Z}_{ij}^{-1}(\mu', \mu, \widetilde{m}_t) \langle h | Q_j(\mu') | h' \rangle \, . \tag{47}$$

The rectangular matrix $\widehat{Z}^{-1}$ can be calculated perturbatively by matching the theory with and without the fictitious top quark. This is exactly the method used to calculate the coefficients $\vec{C}$ in the theory with the physical top quark mass, except that in the physical case one must simultaneously integrate out both the W boson and the top quark. Here we are effectively integrating out the W first and then the fictitious top quark. The results for $\widehat{Z}^{-1}$ at non-leading order can be reconstructed from those computed in refs. [27]–[32]. The running in the 1–2 submatrix is particularly simple (since the operators Q_3–Q_6 do not feed back into Q_1 and Q_2) and one can show that

$$Z_{ij}(\mu' a, g_0^2) = \sum_{k=1,2} \widehat{Z}_{ik}(\mu', \mu) Z_{kj}(\mu a, g_0^2) \, , \tag{48}$$

with $(i, j) = 1, 2$. Here $\widehat{Z}_{ik}$ is the inverse of the 2×2 sub-block of the mixing matrix appearing in eq. (47). This submatrix does not depend on $\widetilde{m}_t$, but only on μ' and μ.

We now choose a value of μ' and vary $\widetilde{m}_t$. The six matrix elements of interest, $\langle h | Q_j(\mu') | h' \rangle$, are obtained by fitting the right hand side of eq. (47) to $\mathcal{M}_i(\mu, \widetilde{m}_t)$ computed numerically as in eq. (44), and using the renormalization matrix $\widehat{Z}^{-1}$ calculated perturbatively. Since the dependence on $\widetilde{m}_t$ is logarithmic, this will not be easy. The procedure is analogous to our use of the x-dependence in sec. 6 to separate the renormalized matrix elements of operators appearing in the weak Hamiltonian. Having determined these renormalized matrix elements, we can insert them into the expression (37) for the effective Hamiltonian. At this point the constraint $M_{_W} \gg \widetilde{m}_t$ can be removed since the Wilson coefficients of the operators appearing in $\mathcal{H}_{eff}^{\Delta S=1}$ can be computed perturbatively for arbitrary values of $\widetilde{m}_t$, including $\widetilde{m}_t = m_t$.

Before discussing the errors involved in this procedure, we make the following observation. In the effective theory where the top quark has been removed, provided that $\mu' \gg \Lambda_{\rm QCD}$, we can evolve the renormalized operators from one scale to another using perturbation theory. In particular we can obtain $\langle h | Q_j(\mu) | h' \rangle$ from $\langle h | Q_j(\mu') | h' \rangle$ for all six operators

using the 6×6 anomalous dimension matrix computed in perturbation theory with the effective Hamiltonian where the top quark has been integrated out. Thus we can directly extract the matrix elements of the operators $\vec{Q}(\mu)$ from $\mathcal{M}_i(\mu, \widetilde{m}_t)$. Note, however, that an accurate determination of the matrix elements of $\vec{Q}(\mu)$ (or of $\vec{Q}(\mu')$) requires that the typical scale, $\Lambda_{hh'}$, of masses and external momenta appearing in the physical process $h' \to h$ be much smaller than $\widetilde{m}_t$. This is because in the matching procedure we neglect terms of $O(\Lambda_{hh'}/\widetilde{m}_t)$.

An alternative method, in the same spirit as the approach followed in sec. 6, is the following. We can avoid the need for any non-perturbative subtraction by separating the two currents and using a fictitious propagating top quark. Thus we directly match $\langle h|T(J_{\rho L}(x)J^\dagger_{\rho L}(0))|h'\rangle_{\rm top}$, where the subscript indicates the presence of the fictitious top, to the formula

$$\langle h|T(J_{\rho L}(x)J^\dagger_{\rho L}(0))|h'\rangle_{\rm top} = \sum_{i=1,6} c_i(x;\mu',\widetilde{m}_t)\langle h|O^{(i)}(\mu')|h'\rangle\,, \tag{49}$$

where

$$\begin{aligned} J_{\rho L}(x)J^\dagger_{\rho L}(0) &= \bar{s}(x)\gamma_\rho(1-\gamma_5)t(x)\bar{t}(0)\gamma_\rho(1-\gamma_5)d(0) \\ &- \bar{s}(x)\gamma_\rho(1-\gamma_5)c(x)\bar{c}(0)\gamma_\rho(1-\gamma_5)d(0)\,. \end{aligned} \tag{50}$$

The coefficients $c_{1,2}(x;\mu',\widetilde{m}_t) \equiv c_{1,2}(x;\mu')$ are the same as those in sec. 6. The coefficients $c_i(x;\mu',\widetilde{m}_t)$, with $i = 3$–6 are complicated functions of the anomalous dimension matrix which can be worked out from the results of refs. [29] and [31] and computed numerically.

Both methods proposed in this section require small enough lattice spacings to accommodate a number of scales. Like the method of sec. 6 their full implementation is likely to require the next generation of supercomputers.

8 Conclusion

In this paper we have suggested a number of new approaches with which to study the $\Delta I = 1/2$ rule using Wilson-like fermions. These methods can also be used for staggered fermions.

In order to obtain the physical $K \to \pi\pi$ amplitude without relying on chiral perturbation theory, or to study decays such as $B \to \pi\pi$, one must learn how to extract information on final state interactions from Euclidean amplitudes. The method of ref. [20] might make this possible, but detailed numerical studies are needed to assess whether it is practical.

The calculation of the CP violating part of $K \to \pi\pi$ amplitudes with Wilson quarks is very difficult. A completely nonperturbative method may require the addition of a fictitious top quark.

We have also reevaluated the method of ref. [3] involving $K \to \pi$ amplitudes. This approach is likely to be more difficult because of the large number of mixing coefficients which have to be determined non-perturbatively. It may however provide complementary information to the results obtained with the $K \to \pi\pi$ method, and a check of the accuracy of chiral relations.

Only numerical studies will be able to confirm or refute our present intuition that the $K \to \pi\pi$ methods are likely to provide the better results in the near future.

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